Physics Factsheet

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Gravitational Orbits and Planets

Factsheet 283: "Key Preparation: Newton´s Laws and Circular Motion" outlined that a centripetal force is necessary to constrain an object to move in a circle. In a planetary orbit, the gravitational force between the object and another massive body provides the necessary centripetal force.

Gravitational attraction

From Newton's 'Law of Gravitation', the gravitational force between two objects is found by the equation

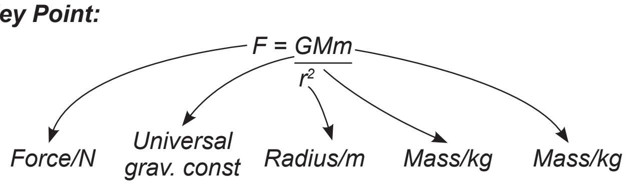
$$\mathbf{F = GMm/r^2}$$

where 'M' and 'm' are the masses of the objects, 'r' is the distance between their centres

and 'G' is the constant of proportionality called the 'Universal Gravitational Constant', commonly referred to as 'big G'. It's not the same as 'g' (little 'g') which is 9.81N/kg – the strength of the Earth's gravitational field at ground level.

$$\mathbf{G = 6.67\times10^{-11}\ Nm^2/kg^2}$$

Key Point:



Worked example: *The Earth has a mass of 6.0 × 10^{24}kg and the Moon has a mass of 7.35 × 10^{22}kg. The distance between them is 3.84 × 10^{8}m. Calculate the gravitational force that Earth exerts on the Moon.*

Answer:

$$F = \frac{GMm}{r^2} = \frac{6.67\times10^{-11}\times6.0\times10^{24}\times7.35\times10^{22}}{(3.84\times10^8)^2} = \mathbf{1.99\times10^{20}N}$$

Centripetal Force

For an orbiting object, the centripetal force necessary is mv^2/r, where m is its mass, v its orbital speed and r the radius of its orbit.

Key Point:

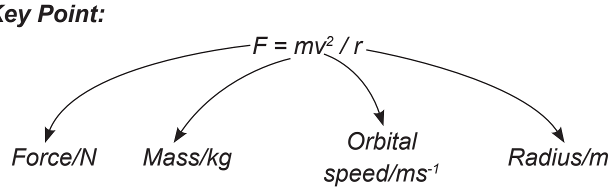
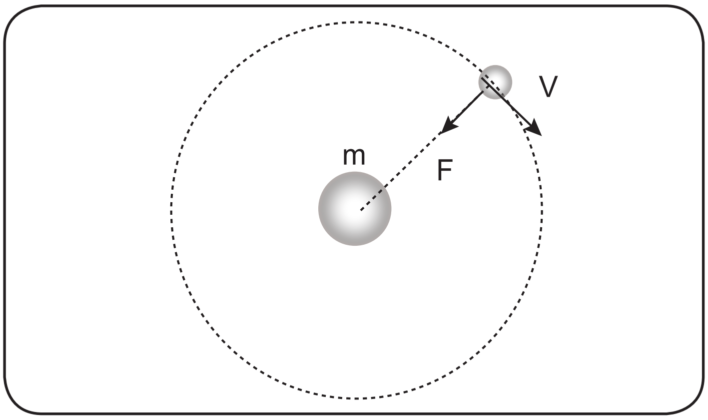


Figure 1



The gravitational attraction provides this force so:

$$GMm/r^2 = mv^2/r$$

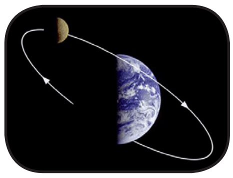
Cancelling the '**m**'s and one of the '**r**'s on the bottom of each term…

Gives: $$\mathbf{v^2 = \frac{GM}{r}}$$

Thus, the orbital speed of an object is dependant only on the mass of the object it is orbiting (M), and the distance separating them (r).

Worked example:

Figure 2



a) What speed is the orbital speed of the Moon around the Earth?

Answer:

$$v^2 = \frac{GM}{r} = \frac{6.67\times10^{-11}\times6.0\times10^{24}}{3.84\times10^8} = 1.04\times10^7$$

so $v = 1021ms^{-1}$

b) At this speed, how long would it take the Moon to complete one orbit.

Answer:

The distance travelled in one circular orbit is effectively the circumference of that orbit.

$C = 2\pi r = 2\times\pi\times3.84\times10^8 = 2.41\times10^9m$

Time for orbit = distance/speed

$= 2.41\times10^9/1021 = 2.36\times10^6s =$ ***27.35 days***

In fact, we can derive an equation for the orbital time 'T' (or orbital period) using these three equations.

$$F = \frac{GMm}{r^2} \qquad F = \frac{mv^2}{r} \qquad v = \frac{2\pi r}{T}$$

Worked example:

a) By substituting the third equation into the second, show that the centripetal force, F, holding an object in orbit can be found using $F = \frac{4\pi^2 r\, m}{T^2}$

Answer: *If* $v = \frac{2\pi r}{T}$ *then* $v^2 = \frac{4\pi^2 r^2}{T^2}$ *so* $F = m \times \frac{4\pi^2 r^2}{T^2}$

$= \frac{4\pi^2 r\, m}{T^2}$

b) Now find an equation for T^2 *in terms of 'G', 'M_1' and 'r'.*

Answer:

$F = G\frac{Mm}{r^2} = \frac{4\pi^2 r\, m}{T^2}$

Cancelling 'm' and collecting 'r's together

$$\frac{GM}{r^3} = \frac{4\pi^2}{T^2}$$

$$\text{So} \quad T^2 = \frac{4\pi^2 r^3}{GM}$$

This equation is known as the Kepler Formula, after Johannes Kepler who used it to explain planetary orbits. It can also be usefully re-arranged to give

$$r^3 = \frac{GMT^2}{4\pi^2}$$

Exam Hint: *Questions often ask you to derive this expression; so it is worthwhile memorising the three equations that are involved and how they link together.*

Try to complete the full process/derivation without the prompts at each stage.

Worked example: *The Sun has a mass of* 1.99×10^{30}*kg. Earth orbits the Sun with an orbital period of 365.25 days. How far is Earth from the Sun?*

Answer:

$r^3 = \frac{GMT^2}{4\pi^2} = \frac{6.67\times10^{-11} \times 1.99\times10^{30} \times (365.25\times24\times60\times60)^2}{39.48}$

$= 3.348\times10^{33}$

$r = 1.50\times10^{11}$m

Orbits round Earth

As we've already seen, the Moon's orbit round Earth is dictated by the orbital-speed or orbital-period, and the orbital-distance. Any object orbiting Earth at that distance would have to be orbiting with the same speed and time-period.

Figure 3

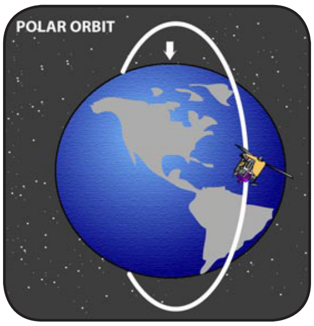
The positions and speeds of the satellites we put in orbit are governed in the same way – their orbital-time and orbital radius have to be consistent with each other.

$$\mathbf{r^3 = \frac{GMT^2}{4\pi^2}}$$

There are two types of satellite orbiting Earth – **low polar satellites** and **geo-stationary** or **geo-synchronous** satellites. Their names derive from the type of orbit they follow.

Low polar satellites

Figure 4



These satellites are pretty much as close to Earth as they could possibly be. They have to be high enough above ground level not to hit mountains or even be slowed down by the atmosphere, but that's it. Ground level is taken as being 6370km from Earth's centre, and the top of the atmosphere is regarded as being 100km above that (although the atmosphere doesn't just stop; it gets thinner and less dense with altitude, and at 100km altitude the density of the atmosphere is effectively zero).

Worked example: *Calculate the orbital time for a Low Polar Satellite, and the necessary orbital speed.*

Answer:

The orbit must be 100km above ground-level, which is 6370km from the centre of Earth. This gives an orbital radius of 6470km.

$T^2 = \frac{4\pi^2 r^3}{GM} = \frac{4\pi^2 \times (6.47\times10^6)^3}{6.67\times10^{-11} \times 6.0\times10^{24}} = \frac{1.07\times10^{22}}{4.0\times10^{14}}$

$= 2.67\times10^7$

$T = 5165\text{s} \equiv$ ***86 mins***

$v^2 = \frac{GM}{r} = \frac{6.67\times10^{-11} \times 6.0\times10^{24}}{6.47\times10^6} = 6.19\times10^7$

so $\mathbf{v = 7860ms^{-1}}$

The key thing about satellites in these low orbits is that they are as close to Earth as a satellite can be. That means they are ideal for aerial photography over a large area, so they are used for mapping (like GoogleEarth/GoogleMaps), photographing weather-systems such as tracking hurricanes, gathering intelligence (identifying potential military installations in hostile or unstable territories) and monitoring the changes in environment – by photographing the ice-caps and the equatorial rain-forests over a period of years, for example, it is possible to see whether their area is changing significantly.

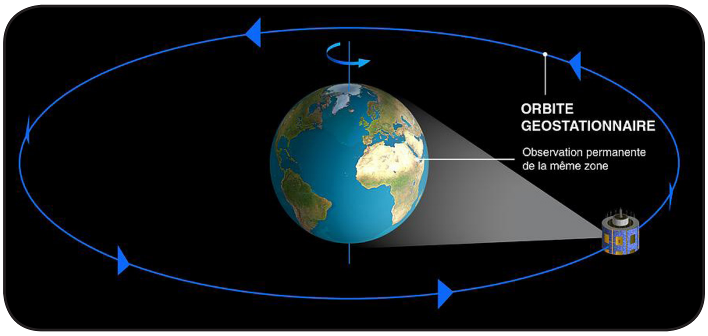
Key Point: *Low orbits are useful for photography*

However, one problem is that any photographs taken by such satellites have to be downloaded from it as it flies overhead, by tracking the satellite's path for the brief time it is overhead every time it comes around.

Geo-stationary orbits

If a satellite has the same time-period as the Earth, it will remain above the same spot on the ground.

Figure 5



Worked example: *At what height must a geostationary satellite be placed above the surface of Earth?*

Answer:

Orbital-time = 24 hours = 24 × 60 × 60 seconds = 86400s

$$r^3 = \frac{GMT^2}{4\pi^2} = \frac{6.67\times10^{-11} \times 6.0\times10^{24} \times (86400)^2}{39.48} = 7.57\times10^{22}$$

$r = 4.23\times10^7$ m

But Earth has a radius of 6370km ≡ 6.37×10^6m, so the height above Earth's surface.

$= 4.23\times10^7\text{ m} - 6.37\times10^6\text{m} = \mathbf{3.59\times10^7 m}$

Key Point: *Geostationary orbits are ideal for communication such as for mobile- phones, satellite TV, GPS and Sat-Nav*

Exam Questions

1) a) The weight w of an object on the Earth can be represented either as

$$w = mg \text{ or } w = \frac{GMm}{r^2}$$

i) Explain the meaning of g and G in these equations.

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(3 marks)

ii) Show that $M = gr^2$

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(1 mark)

iii) Calculate the mass of the Earth to a precision consistent with the data below.

mean radius of the Earth = 6.4×10^6m,

$G = 6.7 \times 10^{-11}\text{Nm}^2\text{ kg}^{-2}$, $g = 9.8\text{ N kg}^{-1}$

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mass of the Earth ....kg

(3 marks)

b) A geostationary satellite is synchronized in orbit with the turn of the Earth.

i) State the time-period for a geostationary satellite.

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(1 mark)

ii) The height of a geostationary satellite in orbit is approximately 36 000 km above the surface of the Earth.

Calculate the radius of a geostationary orbit.

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radius ....m

(1 mark)

iii) Calculate the speed, in kms^{-1}, of a satellite in a geostationary orbit.

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speed ....kms^{-1}

(1 mark)

iv) State a common use for a geostationary satellite.

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(1 mark)

v) Explain why a geostationary orbit is necessary for this use.

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(1 mark)

Answers

1) a) g = gravitational field strength or g = force on 1 kg (on or close to) Earth's surface,
G = gravitational constant **or** G = universal constant relating attraction of any two masses
to their separation/constant in Newton's law of gravitation

ii) $mg = \frac{GMm}{r^2}$ so $g = \frac{GM}{r^2}$ giving $M = \frac{gr^2}{G}$

iii) $M = \frac{gr^2}{G} = \frac{9.8 \times (6.4\times10^6)^2}{6.7\times10^{-11}} = 5.99 \times 10^{24}$ kg

However, notice that all the values used are to 2 significant figures, so therefore the answer should also be, i.e 6.0×10^{24} (kg)

b) i) 1 day / 24 hours / 86400s

ii) 6.4×10^6m + 3.6×10^7m = 4.24×10^7m

iii) $v = \frac{2\pi r}{T} = \frac{2\pi \times 4.24\times10^7}{86400} = 3080$m/s = 3.08kms^{-1}

Don't forget conversion of period to seconds

iv) communication/specific example of communication (eg satellite TV/weather)

v) avoids dish having to track/stationary footprint (position over Earth)

Total 14

Planetary Orbits

The Kepler equation derived before:

$$T^2 = \frac{4\pi^2 r^3}{GM_1}$$

can be applied to planets orbiting their host star, such as in our solar system.

Since the mass of the body being orbited (the Sun) is the a same for all planets in the solar system, a planet's orbital period is determined directly by its distance from the Sun. Equally, the distance of a planet from the Sun is directly linked to its orbital period or year-length, so by making observations and determining a planet's orbit time we can immediately determine the planet's orbital radius.

Worked example: *a) Earth's orbital period is measured as 365.25 days. If the mass of the Sun is taken as being 1.99×10^{30}kg, determine the Earth-Sun distance.*

Answer:

$T^2 = \frac{4\pi^2 r^3}{GM_1}$ therefore $r^3 = \frac{GM_1 T^2}{4\pi^2}$ 365.25 days

$= 365.25 \times 24 \times 60 \times 60 = 3.1558\times10^7$s

$r^3 = 6.67\times10^{-11} \times 1.99\times10^{30} \times (3.1558\times10^7)^2 / (4\times\pi^2) = 3.353\times10^{33}$

$r = 1.497\times10^{11}$ m

b) The mass of the Sun is taken as being 1.99×10^{30}kg. If Saturn's orbital period is observed as being 10759 Earth days, determine the distance of Saturn from the Sun.

Answer:

$T^2 = \frac{4\pi^2 r^3}{GM_1}$ therefore $r^3 = \frac{GM_1 T^2}{4\pi^2}$

10759 days $= 10759 \times 24 \times 60 \times 60 = 9.296\times10^8$ s

$r^3 = 6.67\times10^{-11} \times 1.99\times10^{30} \times (9.296\times10^8)^2 / (4\times\pi^2) = 2.905\times10^{36}$

$r = 1.427\times10^{12}$ m

The Kepler Equation

Using the Kepler equation, astronomers have verified the orbital positions of all the planets with great accuracy, so that we can now predict the motion of the solar system with exceptional reliability e.g when eclipses will take place, down to the minute or better, and where they will best be viewed from; when inner-planets will transit across the face of the Sun, again to the nearest minute or better.

Using planetary data Kepler's equation can be verified:

Table 1

Planet	Distance from Sun (millions of km)	Orbital Velocity (km per second)	Period of Revolution
Mercury	58	48	88 days
Venus	108	35	225 days
Earth	150	30	1 year
Men	228	24	2 years
Jupiter	778	13	12 years
Saturn	1429	10	29 years
Uranus	2875	7	84 years
Neptune	4504	6	165 years

Either by checking that the ratio $r^3:T^2$ is constant for every planet, or plotting a graph of log(R) vs Log(T) to see that the gradient is 2/3, showing that R is proportional to $T^{2/3}$ and therefore R^3 is proportional to T^2.

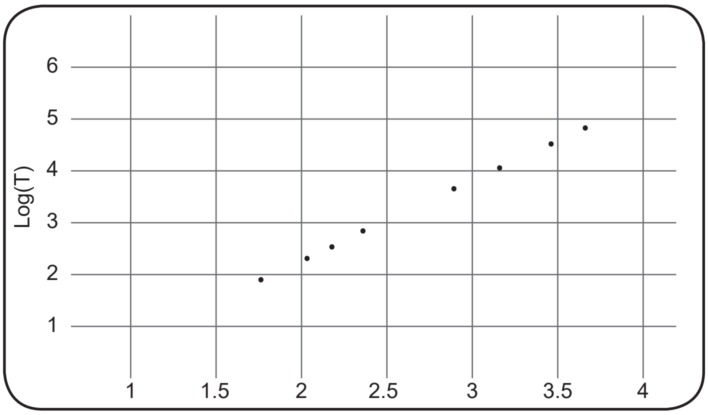
Worked example: *Produce a table of Log(R) and Log(T) then plot a graph to show the gradient confirms r^3 and T^2 are proportional.*

Answer:

Planet	*Log (R)*	*Log (T)*
Mercury	*1.763*	*1.944*
Venus	*2.033*	*2.352*
Earth	*2.176*	*2.563*
Mars	*2.360*	*2.864*
Jupiter	*2.891*	*3.642*
Saturn	*3.155*	*4.025*
Uranus	*3.459*	*4.487*
Neptune	*3.654*	*4.780*

Exam Hint: *Be careful with units. Distances are often quoted in km, but equations use m; times are quoted in days and hours, but equations need seconds.*

Figure 6 Graph of Log(T) vs Log(R) for Planetary Data



Gradient = (4.780-1.944)/(3.654-1.763) = 2.836/1.891 = 1.51

T is proportional to $R^{3/2}$ therefore T^2 is proportional to R^3

Energy considerations

To put either type of satellite in orbit requires a large amount of energy – kinetic energy needed by the rocket to overcome the work done against gravitational potential energy from Earth.

K.E = ½ m v^2 and G.P.E = m g h or m g r

But we cannot use this form of the G.P.E equation – the value for 'g' as 9.81N/kg is not constant over the range of heights involved in launching satellites. A more versatile form of 'g' is required:

$$g = \frac{-GM}{r^2}$$

This means that the gain in Gravitational Potential Energy needed would be found from

$$G.P.E = \frac{m \times -GM \times r}{r^2} = \frac{-GMm}{r}$$

> ***Worked example:*** *Use the equation for Kinetic Energy and Gravitational Potential Energy to derive an expression for the upward velocity needed when launching a rocket.*
>
> ***Answer:***
>
> $\frac{1}{2} m v^2 = \frac{GMm}{r}$
>
> *'m' cancels on each side, giving* $\frac{1}{2} v^2 = \frac{-GM}{r}$
>
> *Therefore* $v = \sqrt{(2GM/r)}$

Notice, here, that 'M' is the mass of the planet being launched from, and 'r' is the distance from the centre of the planet, launching a rocket from a higher position would therefore need a smaller launch velocity. As a result, although 'v' gives the 'escape velocity' from the planet, the velocity needed to launch a satellite would be less than this, for two reasons – firstly, the rocket would be accelerating upwards and would be gaining altitude as it gained velocity, which would reduce the velocity needed. Also, unlike an exploration rocket which would be leaving Earth completely, a satellite is only being put in orbit around Earth so it wouldn't need to overcome all the G.P.E of escaping.

Once in orbit, therefore, the Total Energy of a satellite is the orbital KE it has plus the GPE due to its orbiting height above earth.

Exam Question

2) a) Explain what is meant by the term escape speed.

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(2 marks)

b) Mars has a radius of approximately 3.4×10^6 m and a mass of 6.4×10^{23} kg. Show that the escape speed from Mars is approximately 5 kms^{-1}.

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(3 marks)

c) Explain why a rocket would be able to escape from Mars with an initial speed much less than the escape speed given in part a) ii).

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(3 marks)

Answers

a) minimum or initial speed with which an object escapes the gravitational field

b) Since $\frac{1}{2} m v^2 = \frac{G M m}{r}$ $v^2 = \frac{2 G M}{r}$

$= \frac{2 \times 6.67\times10^{-11} \times 6.4\times10^{23}}{3.4\times10^6} = 2.51\times10^7$

so v = 5011 ms^{-1} **which** approximates to 5 kms^{-1}

c) Any **three** from:

The escape speed assumes an immediate launch and that the object from thereon is unpowered.

During the launch fuel/chemical energy is being transferred into gravitational potential energy.

This can be done at any speed (given sufficient fuel).

Acknowledgements

Earth-Moon picture: http://www.aerospaceweb.org/question/astronomy/moon/orbit.jpg
Satellite orbit: http://upload.wikimedia.org/wikipedia/commons/8/8d/GPS_Satellite_NASA_art-iif.jpg
Polar satellite: http://globalmicrowave.org/content/polar_orbit.jpg
Geostationary Satellite: http://www.sciencephoto.com/image/432247/530wm/C0110804-Geostationary_orbit_diagram-SPL.jpg
Planetary Data: http://www.unawe.org/static/activities/269c20fd-a4d7-42ef-b0bf-e15d7a123c6f/Attachment7_3.jpg

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